Physics 53

Kinematics 1

A child of five can understand this. Send someone to fetch a child of five.
— Groucho Marx

Introduction and overview.

In constructing the description of nature we call *science* it has been found useful to assume that things we observe directly with our senses are actually the combined effects of the behavior of a very large number of objects so small as to be invisible to us. That is, our senses are able to convey to us the gross aspects of things around us, but those aspects are in reality determined by the unobserved behavior of constituent microscopic objects. This "reductionist" assumption about the nature of matter was first introduced largely as a convenient model for mathematical analysis, without any direct evidence for the existence of the small objects. But since the late 1800's we have come to realize that matter really does consist of such things, in the form of molecules, atoms, electrons, neutrons and protons, to which we give the generic name *particles*.

For the purposes of this course we assume only that the particles possess mass, and that they are small enough (by our standards) that, as an approximation, we can say they occupy a single point in space. In the second course other properties they possess, especially their electric and magnetic properties, will be important to us.

This course begins with a description of the behavior of a single particle. At first we are concerned with the appropriate quantitative (i.e., mathematical) description of *where* the particle is located in space at a particular *time*, and how that location changes with time as the particle *moves*. This study is called kinematics of a particle. We then discuss the "influences" that "cause" it to move in particular ways. This leads to dynamics of a particle, for which the relevant general principles are Newton's laws of motion. Influences affecting motion are called interaction forces; we give simple mathematical descriptions of several that occur in everyday situations. Aspects of the *state* of motion itself are expressed through the important concepts of momentum and energy.

Having established the rules for motion of a single particle, we show by mathematical argument how the behavior of systems of more than one particle (perhaps very many) can be accounted for in terms of these same rules. This leads to the discovery of conservation laws of mass, momentum, energy, and angular momentum, specifying conditions under which the *totals* of those quantities for the whole system remain constant in time, no matter how complicated the motions of the individual particles.

The rest of the course describes applications of these general principles. First we examine the behavior of a rigid body, a system of particles tightly bound together. Then we look in some detail at the important interaction called gravity, which Newton showed could account for the motion of satellites — including the planets and their moons — and the tides. Next we consider the simpler aspects of the behavior of fluids, systems in which particles are less tightly bound together so the system has no fixed shape. Then we examine the special repetitive motions called oscillations. This leads to a discussion of a type of collective motion in which particles oscillate locally while passing energy and momentum over longer distances, in the form of waves. Finally we discuss thermal physics, a description in terms of the laws of thermodynamics of the statistical behavior of systems of large numbers of particles in states of thermal equilibrium.

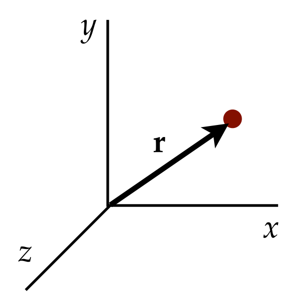
About these notes.

These notes, which are more complete than usual lecture notes but less comprehensive than a textbook, define the *content* of the course. They are self-contained in the sense that every concept, law and formula needed for the course are contained in them. They have only a few worked out examples and few diagrams and pictures, however; for those a textbook is the best reference. So the student should study both the notes and a book, in whichever order seems to work better.

The occasional argument or proof that takes more than a few lines is put at the end of the section. It is helpful to read these arguments and proofs for better understanding of the material, but students will not be expected to reproduce them on exams or quizzes.

Locating a point particle: the position vector.

We start by considering an object so small that we can assume as an approximation it has no spatial size at all, occupying only a single point in space at any particular instant. That point is called its **position**.



To specify the position numerically we use a coordinate system, usually with Cartesian (x, y, z) axes. We choose some point in space to the be origin, and we choose the orientations of the axes. The data specifying the particle's position are the three numbers (x, y, z) giving the coordinates of the position of the particle.

If the particle changes its position and moves to another point, the original coordinate numbers become new ones (x', y', z'). The change in position is called a **displacement** of the particle. We specify it by giving the *differences* between the new and old coordinates: $\left(x' - x, y' - y, z' - z\right)$.

Of course if we had chosen our system of axes differently the numbers specifying the position or the displacement would have been different. But we know intuitively that a change in the position of a particle has its own existence regardless of how we choose our axes. Two people choosing different axes to describe the same particle's change of position will surely agree on the nature of the change, although they will use different sets of numbers.

That the physical nature of the particle's motion cannot depend in any essential way on which coordinate system we choose is an important fact; our description needs to incorporate it. This is the basis for introduction of the ideas of **vectors** and **scalars**.

A **scalar** is a quantity that doesn't change any aspect if we choose a different coordinate system. Simple numbers are scalars. Time is a scalar. Many properties of objects we will study are scalars, such as mass and temperature. It takes only one number (which may be negative) to specify a scalar.

Specification of a **vector**, on the other hand, does change if we choose a different coordinate system, but it changes in a special way: it changes the same way as the specification of the location of a point in space. For example, the position of a particle is specified by its **position vector**, which we denote usually by **r**.

In these notes vectors are denoted by bold type. Some texts use an arrow above the symbol.

To specify the position vector completely requires three numbers, the coordinates (x,y,z). These are called the **components** of the vector. Alternately, we could specify the position by giving the distance one travels to reach it from the origin, and two angles to specify in which direction one goes; that is also three numbers. The distance traveled from the origin, $r = \sqrt{x^2 + y^2 + z^2}$, is called the **magnitude** of the vector **r**. It is a scalar.

Standard notation for the magnitude of a vector is italic type (r), or sometimes magnitude signs ($|\mathbf{r}|$).

The position of a particle is the prototype of a vector, but there are many other quantities we will study that are specified by a vector, including velocity, force and momentum.

If we are describing an actual particle, then because the particle can move the position vector must depend on time. We write the position vector as $\mathbf{r}(t)$. Study of the particle's motion is framed in terms of this function, which we try to estimate, predict or explain. This study is called **kinematics**.

Often it is useful to plot (or at least imagine) the curve in space traced out by the position of the particle as it moves. This is the **trajectory**. Since objects do not jump instantly from one place to another, the trajectory is a continuous curve in space.

Suppose the particle is initially at position given by the vector **r**, and subsequently it is at position given by **r'**. Its displacement vector from the original to the final location is given by $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$, the difference between the position vectors.

This raises the question of how one adds (or subtracts) vectors. The answer is simple: one adds (or subtracts) all the corresponding components. The equation $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$ thus implies the three equations:

$$\begin{aligned} \Delta x &= x' - x \\ \Delta y &= y' - y \\ \Delta z &= z' - z \end{aligned}$$

The equation $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$ is an example of a *vector equation*. A general vector equation has the form $\mathbf{A} = \mathbf{B}$, where **A** and **B** are vectors. As we see from the example above, this kind of equation is actually three separate equations for the components. If those components are (A_x, A_y, A_z) and (B_x, B_y, B_z) then

$$\mathbf{A} = \mathbf{B} \text{ means } \begin{cases} A_x = B_x \\ A_y = B_y \\ A_z = B_z \end{cases}$$

The importance to us of vectors (and scalars) lies in the following: The specific numbers for the components change if we change our system of axes, but they change *in the same way for all vectors*, so equalities among them remain valid. If we express our description of nature in terms of vectors and scalars, validity of that description will *not* depend on our choice of coordinate axes.

We write the statement that the sum of **A** and **B** gives **C** as $\mathbf{C} = \mathbf{A} + \mathbf{B}$, where

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \text{ means } \begin{cases} C_x = A_x + B_x \\ C_y = A_y + B_y \\ C_z = A_z + B_z \end{cases}$$

Besides addition, there are other kinds of combinations. For example multiplication of a vector by a scalar. If c is a scalar then

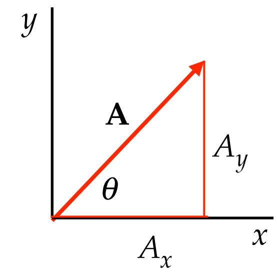
$$\mathbf{B} = c\mathbf{A} \text{ means } \begin{cases} B_x = cA_x \\ B_y = cA_y \\ B_z = cA_z \end{cases}$$

One can think of subtraction of two vectors as multiplication of one of them by -1 and then addition.

One can also multiply vectors, but there are different kinds of products. One product, in which the result is a scalar, will be discussed in the context of work and energy. Another, in which the result is a vector, will be introduced in connection with rotational motion.

There is no definition of division of one vector by another.

Arrow representation.



A common way to show a vector graphically is by means of an arrow. For example, the position vector of a point is represented by an arrow drawn from the origin of the coordinate system to the point. The length of the arrow represents the magnitude of the vector, and angles specify the direction. Other vectors can be similarly represented by arrows.

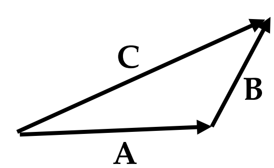
In the 2-D example shown the vector's arrow is the hypotenuse and its components are the sides of a right triangle. This gives us simple and important rules relating the components to the magnitude and direction:

Components of a vector	$A_x = A\cos\theta,\ A_y = A\sin\theta$
Magnitude and direction	$A = \sqrt{A_x{}^2 + A_y{}^2},\ \tan\theta = A_y / A_x$

In these notes items in a bordered table are statements of special importance. These include definitions (as above), fundamental laws, and important properties. The student is expected to know these things and be able to use them on quizzes and exams. Some specific formulas which are widely used are also set in bordered tables; these will be provided on exams in a formula sheet.

The formulas given here apply only in 2-D, where θ is the angle between the vector's direction and the *x*-axis. In 3-D to specify the direction requires two angles. Most of our cases can be handled in 2-D.

Although here we have drawn the arrow representing the vector **A** as starting from the coordinate origin, this is necessary only for position vectors. Other vectors can be represented by arrows drawn as starting wherever appropriate or convenient. Only the direction and length of the arrows are important.



The sum of two vectors is easily represented by arrows. Shown is an example of the sum **C** = **A** + **B**. The arrows for the vectors to be added are placed head to tail. The arrow for the sum is drawn from the tail of the first to the head of the last.

It is easy to show that the result is independent of which vector is called "first", i.e., that **A** + **B** = **B** + **A**. Vector addition is *commutative*. Subtraction can be represented similarly, since multiplication of a vector by –1 reverses the direction of the arrow.

Unit vectors.

One common way to specify a vector makes use of unit vectors, dimensionless vectors of *unit* magnitude, oriented in mutually perpendicular directions. Since any vector can be decomposed into a sum of other vectors, one can write any vector as a sum of the unit vectors, each multiplied by the components of the vector in the respective

directions. One popular set is denoted by $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. These vectors are oriented along the positive (x, y, z) axes respectively. Any vector $\mathbf{A}$ can be written as follows:

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z .$$

This allows one to write all the vector's properties on one line, which is why it is a popular thing to do. It is really no more than listing the Cartesian components.

Unit vectors whose directions can change with time are often useful, and will be introduced in our discussion of motion in a circle.

Motion in one dimension.

Now we are ready to introduce the basic concepts of **kinematics** for a single particle, describing its motion without much inquiry into *why* it moves in a particular way. The question of "why" is the business of **dynamics**.

The simplest motion is that along a straight line. We choose the line of the motion to be the x-axis of a Cartesian coordinate system. The particle's position at any time t is given by a single continuous function $x(t)$.

In vector language, we would say that $\mathbf{r}(t) = (x(t), 0, 0)$, or $\mathbf{r}(t) = \mathbf{i}\, x(t)$.

The first and second derivatives of x with respect to t occur frequently in the analysis, and are given special names: **velocity** and **acceleration**.

Velocity and acceleration (1-D)	Velocity: $v = \dfrac{dx}{dt}$
	Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

The *sign* of the velocity indicates the *direction* of the motion:

- If $v > 0$, x is *increasing* with time, so the particle is moving in the $+x$ direction.
- If $v < 0$, x is *decreasing* with time, so the particle is moving in the $-x$ direction.

The *magnitude* of the velocity is a positive number called the **speed**.

The *sign* of the acceleration indicates how the velocity is *changing*:

- If $a > 0$, v is *increasing* — becoming either more positive or less negative.
- If $a < 0$, v is *decreasing* — becoming either less positive or more negative.

There is no special name for the magnitude of the acceleration.

Finding position from velocity and acceleration.

Since v is the derivative of x, it follows that x is the integral of v. If we know $v(t)$ and x at some initial time t_0, we can find $x(t)$ by integrating:

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t')\,dt'.$$

In the same way, if $a(t)$ and v at t_0 are given, we can find $v(t)$:

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t')\,dt'.$$

The constants $x(t_0)$ and $v(t_0)$ are called the *initial* values. They are often denoted more simply by x_0 and v_0.

Special case: constant velocity.

If v is constant, it can be taken outside the integral in Eq (1) and we find

$$x(t) = x(t_0) + v(t - t_0).$$

This formula can be made to look simpler by choosing the initial time to be $t_0 = 0$, and using the notation x_0 for the initial position:

Constant velocity	$x(t) = x_0 + vt$

Special case: constant acceleration.

For constant a the velocity is easy to find by integration from Eq (2):

$$v(t) = v(t_0) + a(t - t_0).$$

Again taking $t_0 = 0$ and using the notation $v_0 = v(t_0)$ this becomes

Velocity for constant acceleration	$v(t) = v_0 + at$

This function can be substituted into the integral in Eq (1). The result is

Position for constant acceleration	$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$

We can eliminate t between equations for x and v to find a useful formula relating v at any time directly to x at that same time:

Constant acceleration formula (1-D)	$v^2 = v_0^2 + 2a(x - x_0)$

These formulas apply only to the case of constant acceleration.

They are useful because there are many cases of (at least approximately) constant acceleration, but there are many more cases where the acceleration is *not* constant.

Average and Instantaneous Values

The quantities we have defined are called the "instantaneous" velocity and acceleration. An "instant" of time, like a "point" in space, is a mathematical idealization. Instantaneous values of v or a represent mathematical limits as the time interval over which we measure the particle's position or velocity shrinks to zero. But actual measurements take a finite amount of time; what one determines by real measurement is always an average value during some finite time period.

The relation between instantaneous and average values is easily obtained from the mean value theorem in calculus. Applied to the velocity, for example, this theorem says that the average value of v during the time between t_1 and t_2 is given by

$$v_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t')\,dt' .$$

Since the instantaneous velocity in the integrand is $v(t') = dx/dt'$, we can change the variable to x and write

$$v_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{dx}{dt'} dt' = \frac{1}{t_2 - t_1} \int_{x_1}^{x_2} dx = \frac{x_2 - x_1}{t_2 - t_1} .$$

With the usual notation for differences, $\Delta x = x_2 - x_1$, $\Delta t = t_2 - t_1$, we have obtained a simple formula:

Average velocity	$v_{av} = \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{\Delta x}{\Delta t}$

In calculating differences such as Δx, the rule is always *final* value minus *initial* value.

The quantity Δx is the displacement of the particle. The formula says that the average velocity is the displacement divided by the time during which it occurred.

The displacement is *not* necessarily the same as the total distance traveled. For a round trip, ending where it began, the displacement and average velocity are zero, not matter what the total distance traveled or the average speed.

A similar treatment gives the average acceleration:

Average acceleration	$a_{av} = \dfrac{v_2 - v_1}{t_2 - t_1} = \dfrac{\Delta v}{\Delta t}$

These formulas are useful in practice. Many texts start kinematics by introducing the average values for velocity and acceleration, and later introducing the instantaneous values. Partly this is because some students are just beginning their study of calculus, and derivatives are not familiar to them.

Why are Velocity and Acceleration Important?

Despite what the ancients (e.g., Aristotle) thought, it does *not* take an external influence (force) to keep a particle moving with *constant* velocity. On the other hand, it *does* require a force to *change* the velocity in any way. The specific value of the velocity at any instant implies nothing about the forces. But the value of the acceleration *does* give information about the forces. The relation between acceleration and force is the business of dynamics, to be discussed later.

It has not been found necessary to deal directly with time derivatives higher than 2nd in formulating the fundamental laws.